Grading guide, Pricing Financial Assets, June 2012

Problem 1

Let the price of a traded stock, S, be modeled by the geometric Brownian motion

$$dS = \mu S dt + \sigma S dz$$

where μ and σ are constants, and where dt and dz are the standard short hand notations for a small time-step and a Brownian increment.

1. Describe the qualitative characteristics of this model, and discuss it's possible shortcomings.
2. Consider the transformation G of S given by the natural logarithm ($\ln$), i.e. $G(x) = \ln(x)$. Use Ito's lemma to find the process followed by $G(S)$.
3. Suppose that for $t = 0$ the stock price is S_0. What is the expectation of the natural log of the stock price at $t = T \geq 0$?

Answer 1

1. The answer should discuss
   - the drift rate and the volatility
   - the continuous sample paths taken by stock prices
   - the distribution of stock prices and return
   - and that these characteristics often are at odds with empirical findings, where stock price volatility is not constant, stock prices may jump, and return distributions thus will exhibit fatter tails
2. The derivation for the $\ln$-transformation is given in Hull, section 13.7.
3. The expectation
   $$E[\ln S_T] = \ln S_0 + \left(\mu - 0.5\sigma^2 T\right)$$
   follows from the result above.

Problem 2

Suppose certain derivatives have values that depend on a single state variable given by the process

$$\frac{d\theta}{\theta} = m dt + s dz$$

where dz is a Wiener process.

1. Consider two such derivatives, assume a risk free interest rate of r, and use an arbitrage argument to derive and define the market price of ($\theta-$)risk, λ (You may assume that the prices of the derivatives follow geometric Brownian motions).

2. If θ is itself a traded asset, what can we say about the relation between m, s and the market price of risk?

Answer 2

1. Consider two derivatives with prices following two different GBM's with the same underlying Wiener process dz. Use these to form a locally risk free portfolio, and using that this, barring arbitrage, must return the risk free rate you can derive the market price per unit of risk (λ) as the drift of the derivative price in excess of the risk free rate divided by the volatility, i.e. of the form

$$\frac{\mu - r}{\sigma} = \lambda$$

. This is the same for both, hence all such, derivatives (Hull, section 27.1).

2. If θ is itself a traded asset, then (in arbitrage equilibrium)

$$\frac{m - r}{s} = \lambda$$

too.

Problem 3

1. In the Vasicek Model the (instantaneous) short term interest rate r is described by the process:

$$dr = a(b - r)dt + \sigma dz$$

where a, b and σ are constants, and dz a Wiener process. What does this mean for the behavior of the short term interest rate?

2. In this model the short term interest rate shows a predictable pattern. Why is this not necessarily incompatible with (informationally) efficient markets?

3. In the Hull-White Model of the short term interest rate r

$$dr = a(b(t) - r)dt + \sigma dz$$

where a and σ are constants, and dz a Wiener process ($b(t)$ is also written as $\frac{\theta(t)}{a}$). What is the purpose of the extra flexibility compared to the Vasicek Model?

4. Both the above models are one-factor models. What does that mean for the (instantaneous) correlation between changes in interest rates of different maturities (You may illustrate your point taking the Vasicek Model as an example)?

Answer 3

1. The answer should discuss the mean reversion nature of the short rate and the continuous sample paths (Hull, section 30.2).

2. The interest rates are not them selves traded securities.

3. The time-dependent drift term in the Hull-White model is introduced to be able to incorporate a given, initial term structure, making the the values derived from it "arbitrage-free" in relation to the existing securities priced on the current term structure (assuming these to be arbitrage free). This is in contrast to the Vasicek model of the "Equilibrium"-type that put restrictions on the possible initial term structure (as it is a function of the current spot rate only).

   The exact form for the θ-function given the initial term structure is given by Hull i (30.14), while the derivation is a problem (30.14) - this is not considered required for a satisfactory answer to the question.

4. With one-factor models changes in interest rates for different maturities will be perfectly correlated. In the Vasicek-model (or any one factor model of the affine type) the zero rate at t for a maturity of $T - t$ is

   $$R(t,T) = -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r(t)$$

   Since only r is stochastic you can see (or derive) the correlation of 1.